This CD-ROM exhibits many families of quaternionic Julia sets in three dimensional space. Here we give the equations and parameterizations for these families.

Our images are produced by choosing a specific value of θ for each family, and then varying a complex constant $c = R + I$. Thus, a given family depends upon a 2-dimensional array of parameters (R, I). The function that is iterated for a fixed θ (and fixed $c = R + iI$) is

$$f_\theta = e^{-i\theta} z^2 + e^{i\theta} c$$

where z is a quaternionic variable.

The Julia set J_θ is the set of z in 4-space that do not escape to infinity under this repeated iteration. The Julia set

$$\overline{J_\theta} = J_\theta \cap \mathbf{R}^3 = \{(z_0, z_1, z_2) \mid (z_0, z_1, z_2, z_3) \leftrightarrow z_0 + iz_1 + jz_2 + kz_3 \in J_\theta\}$$

is the set that is displayed graphically.

For the record, the recursion has the following form when written out in detail:
Let $z = z_0 + iz_1 + jz_2 + kz_3$, and let $z^2 = a + bi + cj + dk$. Thus

$$a = z_0^2 - z_1^2 - z_2^2 - z_3^2$$

$$b = 2z_0 z_1$$

$$c = 2z_0 z_2$$

$$d = 2z_0 z_3.$$

Then

$$f_\theta = (\cos(\theta) - i\sin(\theta))(a + bi + cj + dk) + (\cos(\theta) + i\sin(\theta))(c_0 + ic_1).$$

Thus

$$\begin{aligned} f_\theta &= \cos(\theta)a + \sin(\theta)b + \cos(\theta)c_0 - \sin(\theta)c_1 \\ &+(-\sin(\theta)a + \cos(\theta)b + \cos(\theta)c_1 + \sin(\theta)c_0)i \\ &+(\cos(\theta)c + \sin(\theta)d)j + (\cos(\theta)d - \sin(\theta)c)k. \end{aligned}$$

The specific distance estimation algorithm is

$$D \approx \frac{|z_n|}{|z_n^{'}|} \ln(z_n)$$

where $z_n = f_\theta^n(z_{initial})$ and $|z_{n+1}^{'}| = 2|z_n||z_n^{'}|$.

Note that since $z_{n+1} = e^{-i\theta} z_n^2 + e^{i\theta} c$, then $z_{n+1}^{'} = 2e^{-i\theta} z_n z_n^{'}$ (by the chain rule), and so $|z_{n+1}^{'}| = 2|z_n||z_n^{'}|$.